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Laminar film condensation

STUART W. CHURCHILL
Department of Chemical Engineering, University of Pennsylvania, 220 South 33rd Street, Philadelphia, PA 19104, U.S.A.

Abstract—The classical solution of Nusselt for laminar film condensation on a vertical surface neglects the effects of the sensible heat and inertia of the condensate, the drag of the vapor and the curvature of the surface. Previous solutions which account for these secondary effects are in forms inconvenient for applications. The present paper presents extended solutions in closed form, and hence applicable for arbitrary parametric values. These solutions are very accurate for large Pr, but for small Pr are restricted to small values of $C_p\Delta T/\lambda$. Both the inertia of the condensate and the drag of the vapor are found to be appreciable for $Pr < 5$.

1. INTRODUCTION

IN 1916 Nusselt [1], in a truly pioneering paper, derived a solution for laminar condensation of a saturated vapor on an isothermal, vertical surface, neglecting the inertia and heat capacity of the condensate and the drag of the vapor. This solution implies invariant physical properties, negligible viscous dissipation and non-rippling flow. He also derived an (erroneous) first-order correction for the effect of the specific heat of the condensate, a correction for a fixed vapor velocity and a correction for the effect of superheat, and discussed the effect of non-condensables semi-quantitatively. He further asserted that the results for a vertical plate were directly applicable for condensation inside and outside vertical tubes and, by neglecting the effect of surface tension, readily modified his basic solution for condensation outside horizontal tubes.

The basic solution derived by Nusselt has been found to be in good functional accord with subsequent experimental data although the observed rates of heat transfer are somewhat higher than the predicted values, owing primarily to rippling of the film.

Most of the idealizations employed by Nusselt have been investigated theoretically and many extensions and improvements have been proposed. Only those directly relevant to the results herein will be noted. Bromley [2] derived an improved first-order correction in closed form for the effect of the heat capacity of the condensate. Rohsenow [3] used an alternative procedure to obtain a similar result. Sparrow and Gregg [4] obtained numerical results for the effects of the inertia (for Pr from 1 to 100) and heat capacity of the condensate (for $J \equiv c_p(T_s - T_w)/\lambda$ up to 2) by solving a boundary-layer model for the liquid phase (neglecting longitudinal molecular transport and the drag of the vapor). Koh *et al.* [5] solved a boundary-layer model for both the condensate and vapor numerically and obtained results for a few discrete values of Pr from 0.003 to 810 and J from 5×10^{-5} to 1.2. This solution revealed that the terms of the model representing vapor drag become increasingly significant as Pr decreases and are in all cases more significant than those representing the inertia of the condensate. Koh [6] solved the same model approximately, using an integral-boundary-layer formulation, and attained results in reasonable agreement with the numerical solution. Chen [7] derived a solution for a modified integral-boundary-layer formulation using a perturbation technique.

The present paper presents an approximate solution of the model of Koh *et al.* [5] which is accurate for most practical values of the parameters and which can be easily evaluated for all such values with a hand-held computer. Analytical solutions are also provided for the effect of curvature on condensation inside and outside vertical tubes.

2. FORMULATION OF MODEL

The boundary-layer model for the conservation of mass, momentum and energy in the stream of condensate on a vertical plate can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(1 - \frac{\rho_v}{\rho}\right) \tag{2}$$

and

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

with

$$u = 0, \quad \delta = 0 \qquad \text{at } x = 0 \tag{4}$$

$$u = v = 0, \quad T = T_w \quad \text{at } y = 0 \tag{5}$$

and

$$-\mu\frac{\partial u}{\partial y} = \rho u\frac{d}{dx}\int_0^\delta u\,dy, \; T = T_s \qquad \text{at } y = \delta. \tag{6}$$

The latter condition on the velocity implies that the viscous drag of the vapor on the liquid is negligible

NOMENCLATURE

A dimensionless constant
a radius of tube [m]
B dimensionless constant
C dimensionless constant
c_p specific heat capacity of condensate [J kg^{-1} K^{-1}]
D diameter of tube [m]
E dimensionless parameter [see equation (22)]
F dimensionless parameter defined by equation (35)
F' dimensionless parameter defined by equation (44)
G dimensionless parameter defined by equation (36)
G' dimensionless parameter defined by equation (45)
g acceleration due to gravity [m s^{-2}]
H $Nu_x\left\{\dfrac{4kv(T_s-T_w)}{\rho g\lambda x^3[1-(\rho_v/\rho)]}\right\}^{1/4}$
h heat transfer coefficient [W m^{-2} K^{-1}]
J $c_p(T_s-T_w)/\lambda$
K $\dfrac{4kv(T_s-T_w)x}{\rho g\lambda a^4[1-(\rho_v/\rho)]}$
k thermal conductivity [W m^{-1} K^{-1}]
Nu_D Nusselt number based on diameter, hD/k
Nu_x Nusselt number based on length, hx/k
M $16Pr/\Delta^4$
Pr Prandtl number, υ/α
r radial distance [m]
T temperature [K]
T_s saturation temperature [K]
T_w surface temperature [K]
U $(T-T_w)/(T_s-T_w)$
u component of velocity in x-direction [m s^{-1}]
v component of velocity in y-direction [m s^{-1}]
x downward distance along surface [m]
y distance from surface [m]

Greek symbols
α thermal diffusivity [m^2 s^{-1}]
γ Δ^4/Pr
Δ $\delta\left[\dfrac{g}{x\alpha\upsilon}\left(1-\dfrac{\rho_v}{\rho}\right)\right]^{1/4}$
η $y\left[\dfrac{g}{x\alpha\upsilon}\left(1-\dfrac{\rho_v}{\rho}\right)\right]^{1/4}$
δ thickness of film of condensate [m]
λ latent heat of condensation [J kg^{-1}]
υ kinematic viscosity [m^2 s^{-1}]
ρ specific density of condensate [kg m^{-3}]
ρ_v specific density of vapor [kg m^{-3}]
Φ $\psi\left\{\dfrac{\upsilon}{g[1-(\rho_v/\rho)]\alpha^3x^3}\right\}^{1/4}$
ψ streamfunction, see equations (8) [m^2 s^{-1}].

and hence that the interfacial shear on the liquid arises primarily from acceleration of the condensing vapor to the velocity of the interface. This approximation is confirmed by the numerical solution of Koh *et al.* [5] who found a negligible dependence of their results on μ_v. The same approximation was made by Chen [7] on the basis of analysis, and indirectly by Koh [6].

The relationship between δ and x is given by the overall energy balance

$$k\left(\frac{\partial T}{\partial y}\right)_{y=0}=\rho\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta[\lambda+c_p(T_s-T)]u\,\mathrm{d}y. \quad (7)$$

Introduction of the streamfunction defined by

$$u=-\frac{\partial\psi}{\partial y},\qquad v=\frac{\partial\psi}{\partial x} \quad (8)$$

and dedimensionalization by the method of Hellums and Churchill [8] identifies a similarity transformation, and permits reduction of the model to the following set of ordinary differential equations:

$$\Phi'''-\frac{1}{Pr}[\tfrac{3}{4}\Phi\Phi''-\tfrac{1}{2}(\Phi')^2]-1=0 \quad (9)$$

and

$$U''=\tfrac{3}{4}\Phi U' \quad (10)$$

with

$$\Phi'=\Phi=U=0 \qquad \text{at } \eta=0 \quad (11)$$

$$4Pr\Phi''=\Phi'(3\Phi-\eta\Phi'),\quad U=1 \qquad \text{at } \eta=\Delta \quad (12)$$

and

$$4JU'\{\Delta\}=-3\Phi\{\Delta\} \quad (13)$$

where

$$\Phi=\psi\left\{\frac{\upsilon}{g[1-(\rho_v/\rho)]\alpha^3x^3}\right\}^{1/4} \quad (14)$$

$$U=\frac{T-T_w}{T_s-T_w} \quad (15)$$

$$\eta=y\left[\frac{g}{x\alpha v}\left(1-\frac{\rho_v}{\rho}\right)\right]^{1/4} \quad (16)$$

and

$$\Delta=\delta\left[\frac{g}{x\alpha v}\left(1-\frac{\rho_v}{\rho}\right)\right]^{1/4}. \quad (17)$$

Also

$$Nu_x = \frac{x}{T_s - T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \left\{\frac{x^3 g[1-(\rho_v/\rho)]}{v\alpha}\right\}^{1/4} U'\{0\} \tag{18}$$

or

$$H \equiv Nu_x\left\{\frac{4kv(T_s - T_w)}{\rho g \lambda x^3[1-(\rho_v/\rho)]}\right\}^{1/4} = (4J)^{1/4} U'\{0\}. \tag{19}$$

Equation (19) implies superficially that $H \to 0$ as $J \to 0$. However $4J/\Delta^4$ is always finite, even for $c_p = 0$, owing to the occurrence of α in the definition of Δ [see equation (17)], and $\Delta U'\{0\}$ is also always finite.

3. SOLUTIONS

Although a general solution for finite heat capacity, inertia and interfacial drag is developed below, solutions in which these effects are neglected individually and collectively will first be derived as a basis for evaluation of their contribution to condensation.

3.1. *Negligible inertia*

As long as the inertia of the liquid is neglected no further approximations beyond that for the drag of the vapor which resulted in equations (6) and (12) are required to obtain the solutions presented below. Hence such solutions are presented separately and first.

Finite heat capacity and finite interfacial drag. Letting $Pr \to \infty$ in equation (9) but not in equation (12) allows for the effect of the acceleration of the condensed vapor up to the velocity of the interface but neglects the effect of the subsequent acceleration of the condensate. Three integrations of the resulting reduced form of equation (9) give

$$\Phi = \frac{\eta^3}{6} + \frac{A\eta^2}{2} + B\eta + C. \tag{20}$$

The boundary conditions on Φ and Φ' for $\eta = 0$ require $B = C = 0$. The boundary condition on Φ for $\eta = \Delta$, i.e. equation (12), requires

$$A^2 + A\left(\frac{\Delta}{2} - \frac{8Pr}{\Delta^3}\right) - \frac{8Pr}{\Delta^2} = 0 \tag{21}$$

whose solution can be expressed as

$$E = \frac{-A}{\Delta} = \frac{1 - M + \sqrt{1 + 6M + M^2}}{4} \tag{22}$$

where

$$M = 16Pr/\Delta^4. \tag{23}$$

The coefficient E ranges from 1/2 to 1 as Pr/Δ^4, and hence Pr, increases from 0 to ∞.

Then

$$\Phi = \frac{\eta^3}{6} - \frac{E\Delta\eta^2}{2} \tag{24}$$

and from equation (10)

$$U'' = \left(\frac{\eta^3}{8} - \frac{3E\Delta\eta^2}{8}\right)U'. \tag{25}$$

Integration then gives

$$U' = U'\{0\}\exp\left(\frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8}\right) \tag{26}$$

and for

$$\left|\frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8}\right| < 1 \tag{27}$$

$$U' = U'\{0\}\left[1 + \left(\frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8}\right) + \frac{1}{2}\left(\frac{\eta^4}{32} - \frac{e\Delta\eta^3}{8}\right)^2 + \cdots\right]. \tag{28}$$

Since the maximum value of η is Δ and E varies from 1/2 to 1, the constraint of equation (27) can be expressed more simply and severely as

$$\Delta^4(4E-1) < 32 \tag{29}$$

and even more severely as

$$\Delta < \left(\frac{32}{3}\right)^{1/4}. \tag{30}$$

Equation (30) proves to be an excessive and hence conservative constraint.

Integrating equation (28) gives

$$U = U'\{0\}\left[\eta + \frac{\eta^5}{160} - \frac{E\Delta\eta^4}{32} + \frac{\eta^9}{18432} - \frac{E\Delta\eta^8}{2048} + \frac{E^2\Delta^2\eta^7}{896} + \cdots\right]. \tag{31}$$

The condition $U\{\Delta\} = 1$ requires

$$U'\{0\} = 1/\Delta\left[1 + \left(\frac{1}{160} - \frac{E}{32}\right)\Delta^4 + \left(\frac{1}{18432} - \frac{E}{2048} + \frac{E^2}{896}\right)\Delta^8 \cdots\right]. \tag{32}$$

Then from equation (13)

$$\frac{4J}{\Delta^4}\left[1 + \left(\frac{1}{32} - \frac{E}{8}\right)\Delta^4 + \left(\frac{1}{2048} - \frac{E}{256} + \frac{E^2}{128}\right)\Delta^8 + \cdots\right] = \left(\frac{3E-1}{2}\right)\left[1 + \left(\frac{1}{160} - \frac{E}{32}\right)\Delta^4 + \left(\frac{1}{18432} - \frac{E}{2048} + \frac{E^2}{896}\right)\Delta^8 \cdots\right] \tag{33}$$

A solution of equation (33) in closed form is possible if the explicit and implicit terms in Δ^{12} and higher in equation (33) are dropped. This solution may be accomplished by specifying arbitrary values of E and $4J/\Delta^4$ and solving the resulting quadratic in Δ^4. Alternatively, a solution can be obtained for specifying E and J by multiplying through by Δ^4 before dropping terms in Δ^{12} and higher. This latter procedure yields:

$$\Delta^4 = \left[1+\frac{8J}{3E-1}\left(\frac{E}{8}-\frac{1}{32}\right)\right](1-\sqrt{14FG})/2F \quad (34)$$

where

$$F = \frac{E}{32}-\frac{1}{160}+\left(\frac{8J}{3E-1}\right)\left(\frac{1}{2048}-\frac{E}{256}+\frac{E^2}{128}\right) \quad (35)$$

and

$$G = \left(\frac{8J}{3E-1}\right)\bigg/\left[1+\frac{8J}{3E-1}\left(\frac{E}{8}-\frac{1}{32}\right)\right]^2. \quad (36)$$

H can be calculated from the resulting value of Δ^4 and the specified values of E and $4J/\Delta^4$ or J using equations (19) and (32). The solution for specified $4J/\Delta^4$ corresponds to

$$4J = \left(\frac{4J}{\Delta^4}\right)\Delta^4. \quad (37)$$

The value of Pr to which these solutions correspond is found by rearranging and representing equation (21) as

$$Pr = \frac{E(2E-1)\Delta^4}{16(1-E)} \quad (38)$$

where E is the specified and Δ^4 the calculated value, respectively.

An *exact* solution of equation (33) for *specified* values of J and Pr is possible by iteration. The required amount of calculation is not significantly greater than the evaluation of the above solution in closed form. The terms of higher order than those given explicitly in equation (33) were not found to be significant for physically representative conditions.

Finite heat capacity but negligible interfacial drag. This condition is established by letting $Pr \to \infty$ in equation (12) or (21) which leads to $E = 1$. Equation (33) is then reduced to

$$\frac{4J}{\Delta^4}\left(1-\frac{3\Delta^4}{32}+\frac{9\Delta^8}{2048}+\cdots\right) = 1-\frac{\Delta^4}{40}+\frac{11\Delta^8}{166128}+\cdots \quad (39)$$

Neglecting the terms in Δ^8 and higher, after multiplying through by Δ^4, corresponding to specified $4J$ rather than specified $4J/\Delta^4$, gives

$$\frac{4J}{\Delta^4} = 1+\frac{3J}{8} \quad (40)$$

hence

$$H = \left(1+\frac{3J}{8}\right)^{1/4}. \quad (41)$$

Equation (40) is analogous to the perturbation for small J derived by Nusselt, [1]. However, he erroneously inverted the temperature distribution with respect to y and thereby obtained

$$H = \left(1-\frac{5J}{8}\right)^{1/4}. \quad (42)$$

Neglecting the terms in Δ^{12} and higher in equation (39) for specified J results in a quadratic expression in Δ^4 which can be solved to obtain

$$\Delta^4 = \frac{G'}{2F'}\left[1-\sqrt{1-\frac{16F'J}{(G')^2}}\right] \quad (43)$$

where

$$F' = \frac{1}{40}+\frac{9J}{512} \quad (44)$$

$$G' = 1+\frac{3}{8}J. \quad (45)$$

Then H follows directly from equation (32) for $E = 1$ and equation (19).

This quadratic solution is slightly more accurate for negligible shear and drag than the approximate solutions derived by Bromley [2] and Rohsenow [3] which are closely represented by

$$H = \left(1+\frac{5}{8}J\right)^{1/4}. \quad (46)$$

The correspondence, except for a sign, between equations (42) and (46) is fortuitous.

Equation (39) can be solved iteratively without neglecting the implicit higher-order terms in the series, but the results differ negligibly from equation (43) for physically realistic conditions.

Negligible heat capacity and drag (the basic solution of Nusselt). For this extreme limiting condition the dimensionless differential model reduces to

$$\Phi''' = 1 \quad (47)$$

$$U'' = 0 \quad (48)$$

$$\Phi' = \Phi = U = 0 \quad \text{at } \eta = 0 \quad (49)$$

$$\Phi'' = 0, \quad U = 1 \quad \text{at } \eta = \Delta \quad (50)$$

and

$$4JU'\{0\} = -3\Phi\{\Delta\}. \quad (51)$$

The solution is

$$\Phi = \frac{\eta^3}{6}-\frac{\eta^2\Delta}{2} \quad (52)$$

$$U = \eta/\Delta \tag{53}$$

$$\Delta = (4J)^{1/4} \tag{54}$$

and

$$H = 1. \tag{55}$$

3.2. *Finite inertia*

The non-linear terms in equation (9) appear to preclude an exact, analytical solution for finite Pr. The evaluation of this term using the velocity distribution of equation (24) for $Pr \to \infty$ (negligible inertia and negligible interfacial drag) gives

$$\Phi''' = 1 - \frac{\eta^2\Delta^2}{8Pr}. \tag{56}$$

Integrating thrice and applying $\Phi'\{0\} = \Phi\{0\} = 0$ gives

$$\Phi = \frac{\eta^3}{6} - \frac{\Delta^2\eta^5}{480Pr} + \frac{A\eta^2}{2}. \tag{57}$$

Finite heat capacity and finite interfacial drag. Finite interfacial drag requires the use of equation (12) as a boundary condition. The result is

$$\frac{\gamma E^2}{8} + \left(1 - \frac{\gamma}{16} + \frac{\gamma^2}{3840}\right)E - \left(1 - \frac{\gamma}{24} - \frac{\gamma^2}{1920} + \frac{\gamma^3}{92160}\right) = 0 \tag{58}$$

where

$$\gamma = \Delta^4/Pr. \tag{59}$$

Equation (57) can then be rewritten as

$$\Phi = \frac{\eta^3}{6} - \frac{E\Delta\eta^2}{2} - \frac{\Delta^2\eta^5}{480Pr}. \tag{60}$$

Equation (10) now becomes

$$U'' = \left(\frac{\eta^3}{8} - \frac{3E\Delta\eta^2}{8} - \frac{\Delta^2\eta^5}{640Pr}\right)U'. \tag{61}$$

Integration gives

$$\ln\left\{\frac{U'}{U'\{0\}}\right\} = \frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8} - \frac{\Delta^2\eta^6}{3840Pr} \tag{62}$$

and for

$$\left|\frac{\Delta^4}{32} - \frac{E\Delta^4}{8} - \frac{\Delta^8}{3840Pr}\right| < 1 \tag{63}$$

$$U' = U'\{0\}\left[1 + \frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8} - \frac{\Delta^2\eta^6}{3840Pr} + \frac{1}{2}\left(\frac{\eta^4}{32} - \frac{E\Delta\eta^3}{8} - \frac{\Delta^2\eta^6}{3840Pr}\right)^2 + \cdots\right]. \tag{64}$$

Then integration with $U\{0\} = 0$ gives

$$U = U'\{0\}\left[\eta + \frac{\eta^5}{160} - \frac{E\Delta\eta^4}{32} - \frac{\Delta^2\eta^7}{26880Pr} + \frac{\eta^9}{18432} + \frac{E^2\Delta^2\eta^7}{896} - \frac{E\Delta\eta^8}{2048} + \cdots\right]. \tag{65}$$

From $U\{\Delta\} = 1$,

$$U'\{0\} = 1/\Delta\left[1 + \Delta^4\left(\frac{1}{160} - \frac{E}{32}\right) + \Delta^8\left(\frac{1}{18432} - \frac{1}{26880Pr} - \frac{E}{2048} + \frac{E^2}{896}\right) + \cdots\right]. \tag{66}$$

From equation (13)

$$\frac{4J}{\Delta^4}\left[1 + \Delta^4\left(\frac{1}{32} - \frac{E}{8}\right) + \Delta^8\left(\frac{1}{2048} - \frac{1}{3840Pr} - \frac{E}{256} + \frac{E^2}{128}\right) + \cdots\right]$$
$$= \left(\frac{3E}{2} - \frac{1}{2} + \frac{\Delta^4}{160Pr}\right)\left[1 + \Delta^4\left(\frac{1}{160} - \frac{E}{32}\right) + \Delta^8\left(\frac{1}{18432} - \frac{1}{26880Pr} - \frac{E}{2048} + \frac{E^2}{896}\right) + \cdots\right]. \tag{67}$$

If γ is specified (arbitrarily), equation (58) which is a quadratic in E can be solved analytically. If terms in Δ^{12} and higher are dropped, equation (67) becomes a quadratic in Δ^4 and can in turn be solved analytically. Finally, H can be calculated from equations (19) and (66). The Prandtl number to which this solution corresponds is equal to Δ^4 divided by the specified γ. However, the iterative solution of equations (58) and (67) does not require significantly more computation, avoids neglecting the higher-order turns in Δ and provides results for prespecified Pr as well as J.

Finite heat capacity but negligible interfacial drag. This reduced case is attained by letting $Pr \to \infty$ in equation (12) or (58). The result is

$$E = 1 - \frac{\Delta^4}{24Pr} \tag{68}$$

which, when substituted in equation (67), gives

$$\frac{4J}{\Delta^4}\left[1 - \frac{3}{32}\Delta^4 + \left(\frac{9}{2048} + \frac{19}{3840Pr}\right)\Delta^8 + \cdots\right]$$
$$= \left(1 - \frac{9}{160}\frac{\Delta^4}{Pr}\right)\left[1 - \frac{\Delta^4}{40} + \left(\frac{11}{16128} + \frac{17}{13440Pr}\right)\Delta^8 + \cdots\right]. \tag{69}$$

Equation (69) can be solved as before, i.e. as a quadratic in Δ^4 if terms in Δ^{12} and higher are dropped, or iteratively in general.

3.3 *Comparison of solutions*

Illustrative calculations were carried out for a wide array of values of J and Pr to provide a basis for comparison of the various solutions described above with each other, with the prior analytical solutions of Nusselt [1], Bromley [2] and Rohsenow [3], and with the numerical solution of Koh *et al.* [5]. Comparisons with the integral-boundary-layer solutions of Koh [6] and Chen [7] were not feasible since numerical values were not provided.

Equations (67) and (58) provide results in close agreement with the numerical solution for all J for $Pr \geqslant 1$ and for small J for small Pr. Indeed, the analytical and iterative results obtained from equations (58) and (67) reveal slight random discrepancies in the numerical solution of Koh *et al.* [5], presumably owing to roundoff errors, and are therefore possibly more accurate. Such discrepancies are however far less than the inherent errors in the general model owing to the neglect of physical property variation, rippling, etc. and hence not of practical significance.

The effects of vapor drag and inertia are both negligible for $Pr \geqslant 5$ hence, all of the approximate solutions, except that of Nusselt, are acceptable for that regime. Conversely, both effects are appreciable and of the same order of magnitude for $Pr < 1$, both decreasing the rate of heat transfer; solutions which neglect either are not then valid.

As Pr decreases below unity and J increases the approximation made herein for the effect of the inertia of the condensate becomes inaccurate, resulting in the overprediction of the rate of condensation.

As also noted by Koh *et al.* [5], Koh [6] and Chen [7], the heat capacity of the liquid increases the rate of heat transfer for $Pr > 1$ but has the opposite effect for $Pr < 1$.

4. EFFECT OF CURVATURE

When a vapor condenses on the outside of a round, vertical tube the effect of curvature is to provide a greater area for flow and a greater area for heat transfer for the same film thickness, thereby resulting in a greater rate of condensation. When a vapor condenses inside a vertical tube these effects are registered conversely. Solutions for these two situations are readily derived for the extreme limiting conditions postulated by Nusselt, including negligible drag, heat capacity and inertia ($Pr \to 0$, $J \to 0$).

Condensation outside a vertical tube

Equation (1) is eliminated and equation (2) is replaced by

$$\frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)+g\left(1-\frac{\rho_v}{\rho}\right)=0 \tag{70}$$

and equation (3) by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)=0. \tag{71}$$

The boundary conditions become

$$u=0,\quad T=T_w \qquad \text{at } r=a \tag{72}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r}=0,\quad T=T_s \qquad \text{at } r=a+\delta. \tag{73}$$

Solution of equations (70) and (71) with these conditions results in

$$u=\frac{ga^2}{2\nu}\left(1-\frac{\rho_v}{\rho}\right)\left[\left(1+\frac{\delta}{a}\right)^2\times\ln\left\{\frac{r}{a}\right\}+\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)\right] \tag{74}$$

and

$$\frac{T-T_w}{T_s-T_w}=\frac{\ln\{r/a\}}{\ln\{1+(\delta/a)\}}. \tag{75}$$

The relationship between x and δ is given by

$$2\pi ak\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=a}=\lambda\rho\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{a+\delta}u\cdot 2\pi r\,\mathrm{d}r. \tag{76}$$

Substituting for T and u from above gives

$$\frac{k(T_s-T_w)}{a\ln\{1+(\delta/a)\}}=\frac{\lambda\rho g a^3}{8\nu}\left(1-\frac{\rho_v}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1+\frac{\delta}{a}\right)^4\left(4\ln\left\{1+\frac{\delta}{a}\right\}-3\right)+4\left(1+\frac{\delta}{a}\right)^2-1\right]. \tag{77}$$

Letting

$$\ln\left\{1+\frac{\delta}{a}\right\}=\frac{\delta}{a}-\frac{1}{2}\left(\frac{\delta}{a}\right)^2+\cdots \tag{78}$$

simplifying and integrating then gives

$$\left(\frac{\delta}{a}\right)^4\left[1+\frac{2}{3}\frac{\delta}{a}+\cdots\right]=\frac{4k(T_s-T_w)\nu x}{\lambda\rho a^4 g[1-(\rho_v/\rho)]}=K. \tag{79}$$

Hence for $\delta/a \ll 1$

$$\frac{\delta}{a}\cong\left(\frac{K}{1+\frac{2}{3}K^{1/4}}\right)^{1/4}. \tag{80}$$

Since

$$h=\frac{k}{T_s-T_w}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=a}$$

$$=\frac{k}{a\ln\{1+(\delta/a)\}}\cong\frac{k}{\delta[1-\frac{1}{2}(\delta/a)]} \tag{81}$$

$$Nu_D=\frac{hD}{k}\cong\frac{2(1+\frac{2}{3}K^{1/4})^{1/4}}{\{1-\frac{1}{2}[K/(1+\frac{2}{3}K^{1/4})]^{1/4}\}K^{1/4}}. \tag{82}$$

The series expansions and the approximation incorporated in this derivation are quite justified since δ/a and K are far less than unity for all practical conditions. The factor

$$(1+\tfrac{2}{3}K^{1/4})^{1/4}/\{1-\tfrac{1}{2}[K/(1+\tfrac{2}{3}K^{1/4})]^{1/4}\}$$

represents the increased rate of heat transfer due to curvature. This correction is significant for $K > 30\times10^{-6}$ or $Nu_D < 22$.

Condensation inside a vertical tube

The above model is changed only by the boundary condition

$$\frac{du}{dr} = 0, \quad T = T_s \qquad \text{at } r = a - \delta \tag{83}$$

and the result is

$$\frac{hD}{k} = \frac{2(1-\frac{2}{3}K^{1/4})^{1/4}}{\{1+\frac{1}{2}[K/(1-\frac{2}{3}K^{1/4})]^{1/4}\}K^{1/4}}. \tag{84}$$

The factor

$$(1-\tfrac{2}{3}K^{1/4})^{1/4}/\{1+\tfrac{1}{2}[K/(1-\tfrac{2}{3}K^{1/4})]^{1/4}\}$$

represents the decreased rate of heat transfer due to internal curvature, which is significant (greater than 5%) for $K > 35 \times 10^6$ or $Nu_D < 26$.

5. SUMMARY AND CONCLUSIONS

Solutions in closed form are presented for the effects of the heat capacity of the condensate, the inertia of the condensate, the drag of the vapor and the curvature of the surface on the rate of laminar, non-rippling condensation of a saturated vapor on a vertical, isothermal surface. Since some of these closed-form solutions do not allow direct specification of the usual independent variables, algebraic equations are also provided which can readily be solved by iteration for specified values of these variables.

These solutions have a greater range of validity than previous solutions in closed form. However, approximations incorporated in these new solutions restrict their applicability for small Pr to small values of $c_p(T_s - T_w)/\lambda$. Most practical applications do fall within their range of validity.

The effects of vapor drag and inertia were both found to be appreciable for $Pr < 5$.

The solutions for the effect of curvature, although in the form of perturbations of the Nusselt solution, are applicable for the more general conditions noted above.

For $Pr > 1$ the heat capacity of the liquid increases the rate of heat transfer as implied by the approximate solutions of Bromley [2] and Rohsenow [3], but has the opposite effect for $Pr < 1$.

The effect of curvature increases the rate of heat transfer significantly for condensation outside a vertical tube only for conditions such that $Nu_D < 27$, and decreases the rate of heat transfer for condensation inside a vertical tube correspondingly.

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CONDENSATION EN FILM LAMINAIRE

Résumé—La solution classique de Nusselt pour la condensation en film sur une surface verticale néglige les effets de la chaleur sensible et de l'inertie du condensat, la trainée de la vapeur et la courbure de la surface. Les solutions qui tiennent compte de ces effets secondaires ont des formes peu adaptées à l'utilisation. Cette étude présente des solutions générales applicables pour des valeurs paramétriques arbitraires. Ces solutions sont très précises pour de grands Pr mais sont limitées aux faibles valeurs de $C_p\,\Delta T/\lambda$ pour des petits Pr.

L'inertie du condensat et la trainée de la vapeur sont appréciables tous les deux lorsque $Pr < 5$.

LAMINARE FILMKONDENSATION

Zusammenfassung—Die klassische Lösung von Nusselt für die laminare Filmkondensation an einer senkrechten Oberfläche vernachlässigt die Einflüsse der fühlbaren Wärme und der Trägheitskräfte des Kondensats, die der Dampfschubkräfte und die der Krümmung der Kondensatfilmoberfläche. Die früheren Lösungen, die sich mit diesen sekundären Einflüssen befassen, sind in der Regel für Berechnungen ungeeignet. Der vorliegende Bericht gibt erweiterte Lösungen in geschlossener Form an, die daher für beliebige Parameterwerte anwendbar sind. Die Lösungen sind sehr genau für große Prandtl-Zahlen, aber beschränkt auf kleine Werte $c_p \cdot \Delta T/\lambda$ für kleine Prandtl-Zahlen. Sowohl die Trägheitskräfte des Kondensats wie auch die Schubkräfte des Dampfes zeigen für $Pr < 5$ beträchtliche Auswirkungen.

ЛАМИНАРНАЯ ПЛЕНОЧНАЯ КОНДЕНСАЦИЯ

Аннотация—В классической задаче Нуссельта о ламинарной пленочной конденсации на вертикальной поверхности пренбрегается зависимостью физических свойств от температуры, влиянием инерционных сил, трением на границе раздела пар-жидкая пленка и кривизной поверхности. Ранее полученные решения, учитывающие эти вторичные эффекты, неудобны для практического использования. В данной работе даются решения в замкнутой форме, применимые для произвольных значений параметров. Эти решения достаточно точны для больших чисел Прандтля, в то время, как для малых ограничиваются небольшими значениями $C_p \Delta T/\lambda$. Найдено, что влияние инерционных сил и трения пара заметны при $Pr < 5$.